STRONG POINT-BLASTS IN A COMPRESSIBLE MEDIUM

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The problem of a strong point-blast in a perfect gas was solved by L.I. Sedov, who found the exact solution for plane, cylindrical and spherical waves [1-3].

L.I. Sedov also formulated the problem of a point-blast in a more general ideal medium, and gave the solution for the particular case of an incompressible fluid [1]. These formulations and results can be used to study the problem of a point-blast in a medium like water.

In the following we obtain self-similar solutions to the problem of a point-blast for three concrete new forms of the equation of state of the ideal medium.

1. In order that the problem of a strong blast in a compressible medium have self-similar solutions, it is sufficient that the internal energy of the medium satisfy

$$\varepsilon(p, \rho) = \frac{p}{\rho_0}\varphi\left(\frac{\rho}{\rho_0}\right) + \text{const} \tag{1.1}$$

where ϕ is an arbitrary function of its argument [1,4]. In this case the following relations hold.

1. The adiabatic equation has the form:

$$p = \Psi(S)\chi\left(\frac{\rho}{\rho_0}\right) \tag{1.2}$$

where $\Psi(S)$ is a certain function of the entropy. Below we develop a mechanical theory independent of the form of $\Psi(S)$. The function $\Psi(S)$ is connected with physical properties of the medium, and can be determined from additional physical investigations. The relation between the functions $\phi(R)$ and $\chi(R)$ is defined by

$$\varphi(R) = \frac{1}{\chi(R)}\left(C + \int\frac{\chi(R)}{R^2}\,dR\right), \qquad \chi(R) = \frac{C}{\varphi(R)}\exp\int\frac{dR}{R^2\varphi(R)} \tag{1.3}$$

where C is an arbitrary constant.

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2. In view of (1.1), the equation of state must have the form:

$$T = \exp\int \frac{dR}{R^2\varphi(R)}\,\Phi\left[p\varphi(R)\exp\left(-\int\frac{dR}{R^2\varphi(R)}\right)\right] \qquad \left(R = \frac{\rho}{\rho_0}\right) \tag{1.4}$$

where Φ is a function related to $\Psi(S)$ through $\Psi'(S)=\rho_0\Phi[\Psi(S)]$.

Let us examine the one-dimensional time-dependent motion of an ideal compressible medium. Because of (1.2), the equations of motion have the form:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0, \qquad \frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial r} + (\nu - 1)\frac{\rho v}{r} = 0 \tag{1.5}$$
$$\frac{\partial}{\partial t}\frac{p}{\chi(\rho/\rho_0)} + v\frac{\partial}{\partial r}\frac{p}{\chi(\rho/\rho_0)} = 0$$

where ν=1,2 and 3 for plane, cylindrical and spherical waves, respectively.

The equations of motion do not contain dimensional constants other than the density ρ_0. The remaining dimensional constants enter only into the boundary conditions. We assume the blast to be concentrated at a point (i.e. the energy of the blast is released instantaneously at the center of symmetry at time $t = 0$), and to be strong (i.e. the pressure, p_1, in the undisturbed medium is negligible compared to the pressure at the shock front). In this case the boundary conditions reduce to conditions at the shock front

$$-\rho_1 c = \rho_2(v_2 - c), \qquad \rho_1 c^2 = p_2 + \rho_2(v_2 - c)^2 \tag{1.6}$$
$$\frac{1}{2}c^2 = \frac{1}{2}(v_2 - c)^2 + \frac{p_2}{\rho_2} + \frac{p_2}{\rho_0}\varphi\left(\frac{\rho_2}{\rho_0}\right)$$

where c is the speed of the shock wave; the subscript 2 denotes quantities behind the shock front; 1 in front of it.

Thus, the problem contains only two quantities with independent dimensions: ρ_0 and E_0, $[\rho_0]=ML^{-3}$, $[E_0]=ML^{\nu-1}T^{-2}$; consequently, as was already stated above, the problem has self-similar solutions. In this case we can look for the velocity, density, and pressure in the form

$$v = \frac{r}{t}V(\lambda), \qquad \rho = \rho_0 R(\lambda), \qquad p = \rho_0\frac{r^2}{t^2}P(\lambda) \qquad \left(\lambda = \frac{r}{r_2}\right) \tag{1.7}$$

where r_2 is the radius of the shock wave:

$$r_2 = \left(\frac{E}{\rho_0}\right)^{1/2\delta} t^{\delta} \qquad \left(\delta = \frac{2}{2+\nu}\right) \tag{1.8}$$

with E a certain constant, with dimension of energy, proportional to the energy of the blast: $E_0 = \alpha E$.

Employing (1.7) and (1.8), we can transform the shock conditions (1.6) to non-dimensional form:

$$R_1\varphi(R_2) = \frac{1}{2}\left(1 - \frac{R_1}{R_2}\right), \qquad V_2 = \delta\left(1 - \frac{R_1}{R_2}\right), \qquad P_2 = \delta R_1 V_2 \tag{1.9}$$

From this we will get an equation connecting V_2 and R_2 at the shock:

$$R_2\varphi(R_2) = \frac{V_2}{2(\delta - V_2)} \tag{1.10}$$

Substituting expressions (1.7) for v, ρ, and p into (1.5), we obtain a system of three ordinary differential equations

$$(\delta - V)\frac{dV}{d\ln\lambda} - \frac{1}{R}\frac{dP}{d\ln\lambda} = V^2 - V + 2\frac{P}{R} \tag{1.11}$$

$$(\delta - V)\frac{d\ln R}{d\ln\lambda} - \frac{dV}{d\ln\lambda} = \nu V \tag{1.12}$$

$$(\delta - V)\frac{d}{d\ln\lambda}\ln\frac{P}{\chi(R)} = 2(V - 1) \tag{1.13}$$

These equations have two independent integrals: the energy integral and the adiabatic integral [1]. In view of the shock conditions (1.9), and of the second of relations (1.3), these integrals take the form:

$$P = \frac{(\delta - V)RV^2}{2[V - (\delta - V)R\varphi(R)]}, \qquad \lambda = \left[\frac{K\chi(R)}{R(\delta - V)P}\right]^{1/\delta} \tag{1.14}$$

where

$$K = \frac{R_2(\delta - V_2)P_2}{\chi(R_2)} \tag{1.15}$$

i.e. K is a constant, chosen to make λ equal to one on the shock wave. If $R(V)$ is known, the integrals (1.14) give two equations for the determination of the two quantities, P and λ, as functions of V. Eliminating λ from equations (1.11) and (1.12), we have

$$\nu V\left[(\delta - V) - \frac{1}{R}\frac{dP}{dV}\right] = \left(V^2 - V + \frac{2P}{R}\right)\left[(\delta - V)\frac{d\ln R}{dV} - 1\right] \tag{1.16}$$

Replacing P in (1.16) by its expression in terms of R and V from the energy integral (1.14), we get the following equation:

$$\frac{dR}{dV} = \frac{R^2\varphi(R)}{(\delta - V)} \times \\ \times \frac{\{V[(1-\nu)V - \delta] - (\delta - V)[(1-\nu)V + {}^1/_2(\nu - 2)\delta]R\varphi(R)\}}{\{{}^1/_2\nu V^2[1 - R^2\varphi'(R)] + V[V - (\nu+1)\delta]R\varphi(R) + (1 - V)(\delta - V)R^2\varphi^2(R)\}} \tag{1.17}$$

In this way, the problem is reduced to integrating (1.17), after which P and λ can be found from (1.14).

The behavior of the solution of the system of equations (1.17) and (1.14) depends essentially on the character of the singularities of equation (1.17), which in turn is determined by the analytic properties of the function $\phi(R)$ in the neighborhood of the singularities of equation (1.17).

For the problem of a strong blast in a gas, the function $\phi(R)$, is determined by the relation

$$\varphi(R) = \frac{1}{(\gamma - 1)} \frac{1}{R} \tag{1.18}$$

where γ is the adiabatic exponent.

The exact solution of (1.17), under the condition (1.18), was found by L.I. Sedov [1]. In this case the function $\ln R$ is found by a quadrature; for motion with spherical symmetry, we have

$$R = C\left(\frac{5\gamma}{2}V - 1\right)^{\frac{3}{2\gamma+1}}\left(1 - \frac{5}{2}V\right)^{\frac{2}{\gamma-2}} \left[1 - \frac{(3\gamma-1)}{2}V\right]^{\frac{13\gamma^2-7\gamma+12}{(2-\gamma)(3\gamma-1)(2\gamma+1)}} \tag{1.19}$$

The shock conditions (1.9) give

$$V_2 = \frac{4}{5(\gamma+1)}, \qquad R_2 = \left(\frac{\gamma+1}{\gamma-1}\right) R_1, \quad P_2 = \frac{8}{25(\gamma+1)} R_1$$

From equation (1.19) it can be seen that all the integral curves in the RV plane are similar to each other. The solution of the problem depends essentially on the parameter γ.

If the function $\phi(R)$ has a more general form than given in (1.18), the solution of the problem, given by each integral curve, is determined by the parameter $R_1 = \rho_1/\rho_0$. The first of the shock conditions (1.9) gives

$$R_1 = \frac{R_2}{1 + 2R_2\varphi(R_2)} \tag{1.20}$$

For any function $\phi(R)$, the parameter R_1 depends explicitly on the parameter R_2; consequently we will write the expressions for the velocity, density and pressure at the shock front in terms of R_2:

$$\rho_2 = \rho_1[1 + 2R_2\varphi(R_2)], \qquad v_2 = \frac{2cR_2\varphi(R_2)}{1 + 2R_2\varphi(R_2)}, \qquad p_2 = \frac{2\rho_1 c^2 R_2\varphi(R_2)}{1 + 2R_2\varphi(R_2)} \tag{1.21}$$

Here

$$c = \delta\left(\frac{E}{\rho_0}\right)^{\frac{\delta}{2}} t^{-\frac{\nu\delta}{2}} = \delta\sqrt{\frac{E}{\rho_0}}\, r_2^{-\frac{\nu}{2}}$$

In the formulas obtained above there occurs the constant E, which can be expressed in terms of the blast energy E_0 (which, in the present formulation, is equal to the total energy of the disturbed medium). If the solution can be continued to the center of symmetry, then

$$E_0 = \sigma_\nu \int_0^{r_2} \left[\frac{\rho v^2}{2} + \frac{p\rho}{\rho_0}\varphi\left(\frac{\rho}{\rho_0}\right)\right] r^{\nu-1}\,dr \tag{1.22}$$

where $\sigma_\nu = 2(\nu-1)\pi + (\nu-3)(\nu-2)$.

If at the center of the blast there is created an expanding cavity of radius r_*, enclosing a vacuum, we get the following formula instead of (1.22):

$$E_0 = \sigma_\nu \int_{r_*}^{r_2} \left[\frac{\rho v^2}{2} + \frac{p\rho}{\rho_0}\varphi\left(\frac{\rho}{\rho_0}\right)\right] r^{\nu-1}\,dr \tag{1.23}$$

Making use of (1.7) and (1.8), we can transform the formulas (1.22) and (1.23) to dimensionless form:

$$\alpha = \sigma_\nu \int_0^1 R\left[\frac{V^2}{2} + P\varphi(R)\right]\lambda^{\nu+1}\,d\lambda \tag{1.24}$$

$$\alpha = \sigma_\nu \int_{\lambda_*}^1 R\left[\frac{V^2}{2} + P\varphi(R)\right]\lambda^{\nu+1}\,d\lambda \quad (E_0 = \alpha E) \tag{1.25}$$

where $\lambda_* = r_*/r_2$. In the problems examined below, the constant α is computed from formulas (1.24) and (1.25).

2. It is easy to verify that for any function $\phi(R)$, equation (1.17) has the solution:

$$V = \frac{\delta R\varphi(R)}{1 + R\varphi(R)} \tag{2.1}$$

At each point of the integral curve (2.1), we have $P = \infty$, $\lambda = 0$.

Equations (1.14) and (1.17) also have the following two solutions:

$$R - \text{arbitrary},\ V = \delta,\ P = 0,\ \lambda = \infty \tag{2.2}$$

$$V - \text{arbitrary},\ R = 0,\ P = 0 \tag{2.3}$$

(the value of λ in formula (2.3) depends on the behavior of the function $\phi(R)$ in the neighborhood of the point $R = 0$). From this it follows that in the RV plane, the integral curves for which $P > 0$ are confined between the integral curve (2.1) and the lines $V = \delta$ and $R = 0$, which give the solutions (2.2) and (2.3).

If the function $\phi(R)$ has the form $\phi(R) = (R-1)^k f(R)$, where $k > 0$,

$f(1) \neq 0$, the system (1.14), (1.17) has another solution:

$$R = 1, \qquad P = \tfrac{1}{2} V(\delta - V), \qquad \lambda = 0 \text{ or } \infty \tag{2.4}$$

(if $k = 1$ and $f(1) = 0$ simultaneously, the (2.4) is not a solution). In that case, if $f(1) > 0$, the integral curves for which $\phi(R) > 0$ are confined between the integral curve (2.1) and the lines (2.2), (2.4).

Since the internal energy of the medium is zero at every point of the integral curve $R = 1$ ($\phi(R) = 0$), the formulas (2.4) give the solution to the problem of a strong blast in an incompressible fluid, which was found by L.I.Sedov [1]. The dimensional quantities, velocity, density, and pressure are determined in the case of spherical symmetry, by the formulas

$$v = \frac{2}{5}\left(\frac{E'}{2\pi\rho_0}\right)^{\frac{3}{5}} \frac{t^{\frac{1}{5}}}{r^2}, \qquad \rho = \rho_0 \tag{2.5}$$

$$p = \frac{2}{25}\rho_0\left(\frac{E'}{2\pi\rho_0}\right)^{\frac{2}{5}} t^{-\frac{6}{5}} \frac{r_*}{r}\left[1 - \left(\frac{r_*}{r}\right)^3\right], \qquad r_* = \left(\frac{E'}{2\pi\rho_0}\right)^{\frac{1}{5}} t^{\frac{2}{5}}$$

where E' is a certain constant, proportional to the kinetic energy of the fluid $E'_0 = 4/25\, E'$, and r_* is the radius of the empty sphere, which expands as time goes on.

3. We will now find the curve of weak discontinuities in the RV-plane; on this curve the particle speed relative to the speed of the shock wave is equal to the sound speed, i.e.

$$dp/d\rho = (v - c)^2$$

From this, making use of (1.7) and (1.8), the energy integral (1.14), the adiabatic equation (1.2), and the relation (1.3) between the functions $\phi(R)$ and $\chi(R)$, we get

$$V = \frac{\delta R\varphi(R)\left[1 + 2R\varphi(R) \pm \sqrt{2R^2\varphi'(R) - 1}\right]}{1 - R^2\varphi'(R) + 2R\varphi(R)(1 + R\varphi(R))} \tag{3.1}$$

If $2R^2\,\phi'(R) < 1$ for all values of R, there does not exist a real curve of weak discontinuities for the given function $\phi(R)$.

The variable parameter λ in the function V attains an extreme value along the curve (3.1). Consequently, for motion along an integral curve, a continuous transition across curve (3.1) is impossible; the transition can be effected only be means of a shock wave.

4. In the present section and in the two following ones, we will

investigate the spherically symmetric case.

At the present time there does not exist a universally accepted adiabatic equation of state for water. Usually this equation is taken in the form

$$p = \Psi(S)\left(\rho^{\varkappa} - \rho_0^{\varkappa}\right) \tag{4.1}$$

with the value of κ close to 7 (e.g., see [5],.

It is always possible to choose the constant ρ_0 in such a way that (4.1) yields

$$\chi(R) = R^{\varkappa} - 1 \tag{4.2}$$

It can be shown that if the adiabatic equation has the form (1.2), and the equation of state the form (1.4), then the expression for the internal energy is given by (1.1), and the relations (1.3) hold. Taking into account (4.2), the first of relations (1.3) then given an expression for $\phi(R)$:

$$\varphi(R) = \frac{R^{\varkappa} + C_1 R + \varkappa - 1}{(\varkappa - 1) R (R^{\varkappa} - 1)} \tag{4.3}$$

where C_1 is an arbitrary constant. Setting $C_1 = -\kappa$, we have

$$\varphi(R) = \frac{R^{\varkappa} - \varkappa R + \varkappa - 1}{(\varkappa - 1) R (R^{\varkappa} - 1)} \tag{4.4}$$

In the neighborhood of the point $R = 1$

$$\varphi(R) = \tfrac{1}{2}(R - 1)\left[1 - \tfrac{1}{6}(\varkappa + 7)(R - 1) + \ldots\right] \tag{4.5}$$

and in the neighborhood of $R = \infty$

$$\varphi(R) = \frac{1}{(\varkappa - 1) R}\left[1 - \varkappa\left(\frac{1}{R}\right)^{\varkappa - 1} + \cdots\right] \tag{4.6}$$

From results of experiments [6,7] which give the density and the temperature of water under high pressure, we obtained the values $\kappa = 20/3$ for the exponent in formula (4.2), and $\rho_0 = 0.93894$ g/cm^3 for the density ρ_0 occurring in (4.1).

For simplicity, let us assume that $\phi(R)$ has the form

$$\varphi(R) = \frac{R - 1}{2R^2} \tag{4.7}$$

In the neighborhood of the point $R = 1$, the function (4.7) has the following asymptotic behavior:

$$\varphi(R) = \tfrac{1}{2}(R - 1)\left[1 - 2(R - 1) + \ldots\right] \tag{4.8}$$

In the expansions (4.8) and (4.5), the first terms coincide.

The function (4.4) can be approximated by (4.7) with a known accuracy.

Now let us examine the problem of a strong blast for the function (4.7). The system of integral curves of (1.17) is shown in Fig. 1. (The heavy line in Figs. 1, 5 and 7 represents the curve (1.10).)

The curve of weak discontinuities (3.1) passes through only one point ($V = 0$, $R = 1$) in the domain of values of (R, V) under examination. The equation (1.17) has five singular points in this domain.

The point $C(V = 0.4,\ R = 1)$ is a node. In the neighborhood of this point we have

$$R = 1 + C_1 (0.4 - V)^2, \qquad P = 0.5V (0.4 - V), \qquad \lambda = (5\, C_1 K)^{0.2} \tag{4.9}$$

Here C_1 is an arbitrary constant, and

$$K = \frac{(0.4 - V_2)\, P_2}{R_2 (R_2 - 1)} \tag{4.10}$$

From the asymptotic form of (4.9) it is clear that the point C corresponds to the boundary between the moving fluid and the cavity.

The point D ($V = 0.4$, $R = \infty$) is a saddle point. Only one integral curve, $V = 0.4$, passes through it. The point B ($V = 2/15$, $R = \infty$) is also a saddle point. The only integral curve passing through it is (2.1), which, for the function $\phi(R)$ in question, is determined by the equation

$$R = \frac{0.4 - V}{0.4 - 3V} \tag{4.11}$$

The point F ($V = 0.25$, $R = \infty$) is a node.

The point A ($V = 0$, $R = 1$) is a compound singularity; here we have both a node and a saddle point.

The asymptotic formula for R as a function of V has the form

$$\frac{V}{0.4 - V} - \frac{(R - 1)}{2R} = C \left(\frac{R - 1}{2R}\right) \left(\frac{R - 1}{R} - \frac{V}{0.4 - V}\right)^{8/3} \tag{4.12}$$

Making use of the integrals (1.14), we can find the values of P and λ in the neighborhood of point A:

$$\begin{aligned} P &= \frac{V^2}{C (R - 1)} \left[\frac{R - 1}{R} - \frac{V}{0.4 - V}\right]^{-5/3} \\ \lambda &= \left[\frac{2.5KC (R - 1)^2}{V^2}\right]^{1/5} \left(\frac{R - 1}{R} - \frac{V}{0.4 - V}\right)^{1/3} \end{aligned} \tag{4.13}$$

From these formulas we obtain different asymptotic behavior for different integral curves. The asymptotic formulas (4.12) and (4.13) for the integral curves entering point A, and therefore corresponding to a node, can be written in the form

$$R = 1 + 5V + 37.5V^2 + C_2 V^{8/3}$$
$$P = -\frac{1}{C_2} V^{-8/3}, \qquad \lambda = (-12.5KC_2)^{1/3} V^{1/3} \tag{4.14}$$

(C_2 is a new arbitrary constant).

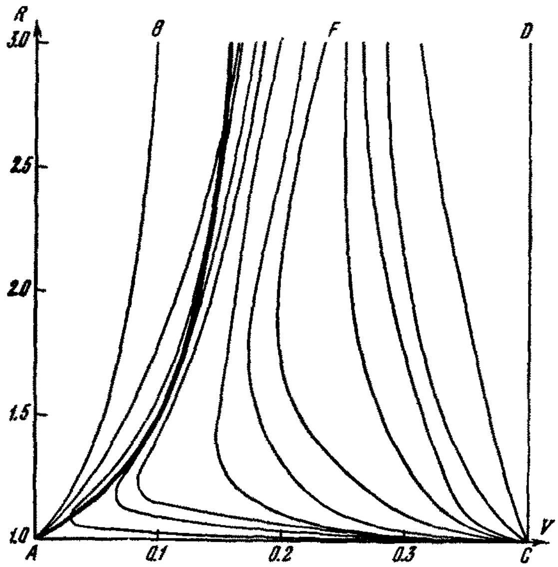

Fig. 1.

We note that the coefficient of V in the expression for R depends on the analytic properties of the function $f(R)$ in the neighborhood of $R = 1$, if the function $\phi(R)$ is determined by the equation $\phi(R) = ½(R-1)[1 + f(R)]$, with $f(1) = 0$.

It can be seen from (4.14) that the point A corresponds to the center of symmetry. The curves entering the point A give a solution to the problem, which can be continued to the center of symmetry, where the velocity vanishes, while the pressure and the density remain finite.

Furthermore, the integral curves belonging to the saddle point enter the point C, and are separated from the curves entering point A by a curve tangent to curve (1.10) at point A; curve (1.10) gives the condition at the shock, and in the present case has the form:

$$R_2 = \frac{0.4 - V_2}{2(0.2 - V_2)} \tag{4.15}$$

For the separating curve, the asymptotic formulas in the neighborhood of point A have the form:

$$R = 1 + 2.5V + 13.75V^2, \qquad P = V(0.4 - V), \qquad \lambda = 0.696 \tag{4.16}$$

At the center of symmetry there is formed a region of fluid at rest, expanding as time goes on, where the density is constant and equal to ρ_0, and the pressure vanishes.

Now let us examine those integral curves which give the solution to the problem of a strong blast. As we have noted above, the point A corresponds to the center of symmetry ($\lambda = 0$), and the point C (λ = const, $p = 0$) to the boundary of the cavity. From the point ($V = 0$, $R = R_1$), which characterizes the state of the fluid prior to the blast, it is possible to get to an integral curve entering either A or C only by means of a jump at point (V_2, R_2) of the curve (4.15). All integral curves leaving points A and C (except for curves $V = 0.4$ and (4.11)), enter point F. At the point ($V = 0.1$, $R = 1.5$) one of these curves is tangent to curve (4.15).

All curves which lie between this integral curve and curve (4.11), cross curve (4.15). Consequently, each one of these curves, characterized by a given value of the parameter R_1, gives the solution to some problem of a strong blast in a medium specified by (4.7). Furthermore, the parameter R_1 takes on all values between 1 and ∞.

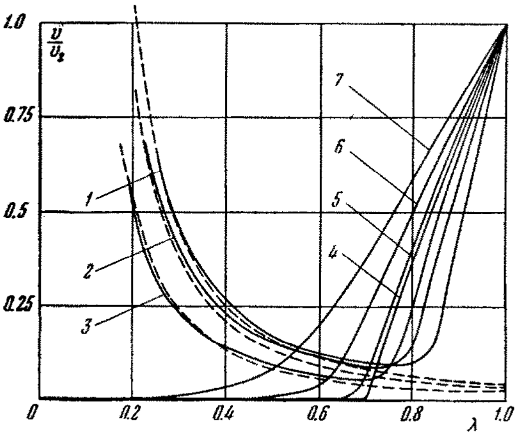

Fig. 2.

Graphs of the functions v/v_2 and ρ/ρ_2, p/p_2, are shown in Figs.2 and 3; curves labeled 1 correspond to $R_1 = 1.0667$, 2 – 1.1250, 3 – 1.1910, 4 – 1.2921, 5 – 1.3333, 6 – 1.5625, 7 – 2.2042.

Let us examine somewhat more closely the behavior of these functions for different values of R_1. For $R_1 = 1$ we get the integral curve

$$R = 1, \qquad P = 0.5V(0.4 - V) \quad \lambda = 0 \tag{4.17}$$

The formulas (4.17) give the solution to the problem of a strong blast in an incompressible medium [1]. In view of (1.7) and (1.8), we get formulas (2.5) from (4.17) if we set $V = 0.4(r_*/r)^3$, where r_* is the radius of the cavity (2.5). Corresponding to this, making use of (1.21) and (4.17), we get from (2.5): $v/v_2 \longrightarrow \infty$, $p/p_2 \longrightarrow \infty$, $\rho/\rho_2 \longrightarrow 1$.

The integral curves corresponding to the values $0 < R_1 < 1.2921$ leave the point C and enter point F, crossing the curve (4.15) at two points; to each value of R_1 there corresponds a point of intersection. These curves give the solution to the problem of a blast with spherical cavity expanding from the center (curves 1, 2, and 3 of Figs.2 and 3).

In the neighborhood of the cavity the fluid behaves as if it were incompressible. The compressibility is significant only in the vicinity of the shock wave. The ratio of the blast energies for incompressible and compressible fluids is equal to

$$\frac{E_0'}{E_0} = \frac{0.32\pi\lambda_*^5}{\alpha} \tag{4.18}$$

where λ_* is the dimensionless radius of the cavity ($\lambda_* = r_*/r_2$), and $\alpha = E_0/E$. The ratio (4.18) decreases from one to zero as R_1 varies from 1 to 1.2921.

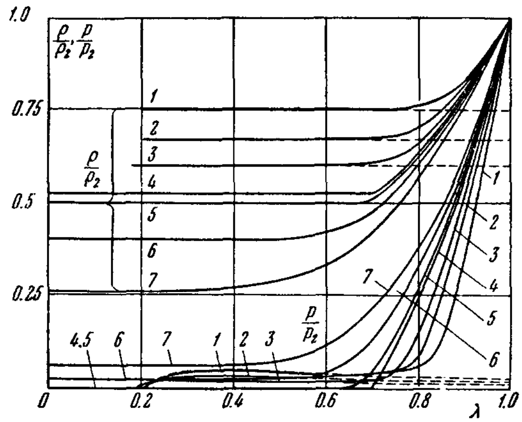

Fig. 3.

Consequently, near the cavity the fluid can be considered incompressible, with its blast energy given by (4.18).

The dashed curves of Figs.2 and 3 correspond to curves 1, 2 and 3 for the case of an incompressible fluid. In the vicinity of the cavity, where they give good approximations to the values of v/v_1, ρ/ρ_2, and p/p_2, these curves were used as asymptotes of the correct ones.

The separating curve (curve 4 in Figs. 2 and 3) corresponds to $R_1 = 1.2921$. As λ varies from 1 to 0.696, the density decreases to ρ_0, and the pressure and velocity to zero; for $\lambda = 0.696$ the fluid goes over into a state of rest by means of a weak shock, and the curves therefore have a corner at this point. For $\lambda \leqslant 0.696$ the velocity and the pressure vanish, and the density is equal to ρ_0; corresponding to this, at any time the mass of the fluid behind the shock front is equal to the mass initially in this volume. Thus the separating curve gives the solution to the problem of a blast with a region of fluid at rest, expanding from the center according to the law $r = 0.696\ (E/\rho_0)^{1/5}\ t^{2/5}$.

The curves entering point A and tangent at this point to (4.11), give the solution to the problem of a blast for $1.2921 < R \leqslant \infty$ (curves 5, 6, 7 of Figs. 2 and 3), corresponding to which the velocity vanishes at the center of symmetry, and the pressure and density are finite.

If $R_1 \to \infty$ (the blast takes place in a fluid of infinite density), the speed of the shock front goes to zero, and the density behind it becomes infinite.

Fig.4 shows the graphs of the functions $\alpha = E_0/E$ and E_0' / E_0 ((4.18)) as functions of R_1.

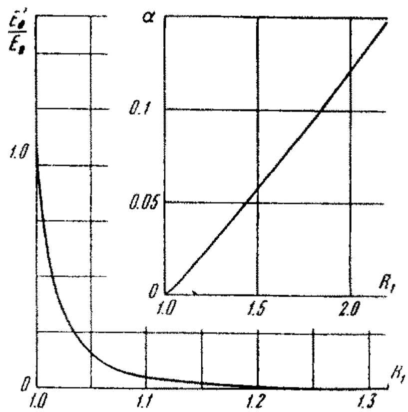

Fig. 4.

5. Let us examine the case when the function $\phi(R)$ is given by the formula

$$\varphi(R) = \frac{(\gamma - 1) R^2 + (\gamma_1 - 1) a^2}{(\gamma - 1)(\gamma_1 - 1) R (a^2 + R^2)} \tag{5.1}$$

where γ, γ_1 and a are constants, with γ and γ_1 greater than one.

Making use of the second of relations (1.3), we can find $\chi(R)$:

$$\chi(R) = C(a^2 + R^2) R^{\gamma} \left[1 + \left(\frac{\gamma - 1}{\gamma_1 - 1}\right)\left(\frac{R}{a}\right)^2\right]^{\frac{\gamma_1 - \gamma}{2} - 1} \tag{5.2}$$

We can write down the first few terms of the expansions of $\phi(R)$ and $\chi(R)$ in the neighborhood of $R = 0$:

$$\varphi(R) = \frac{1}{(\gamma - 1) R}\left\{1 + \left(\frac{\gamma - \gamma_1}{\gamma_1 - 1}\right)\left(\frac{R}{a}\right)^2 + \cdots\right\}$$

$$\chi(R) = CR^{\gamma}\left\{1 + \left(\frac{\gamma_1 - \gamma}{\gamma_1 - 1}\right)\frac{(\gamma+1)}{2}\left(\frac{R}{a}\right)^2 + \cdots\right\}$$

In the neighborhood of $R = \infty$ we get the following expansions:

$$\varphi(R) = \frac{1}{(\gamma_1 - 1)R}\left\{1 + \left(\frac{\gamma_1 - \gamma}{\gamma - 1}\right)\left(\frac{a}{R}\right)^2 + \cdots\right\}$$

$$\chi(R) = C\left(\frac{\gamma_1 - 1}{\gamma - 1}\right)^{2+\gamma-\gamma_1} a^{\gamma}\left(\frac{R}{a}\right)^{\gamma_1}\left\{1 + \frac{(3-\gamma_1)}{2}\left(\frac{\gamma - \gamma_1}{\gamma - 1}\right)\left(\frac{a}{R}\right)^2 + \cdots\right\}$$

Thus, the expression for the internal energy and the adiabatic equation coincide, up to quantities of second order, with those for a gas with adiabatic exponent γ and γ_1 respectively.

The exact solution of (2.1) and the shock condition (1.10) are given respectively by

$$R = a\sqrt{\left(\frac{\gamma_1 - 1}{\gamma - 1}\right)\left(\frac{0.4 - \gamma V}{\gamma_1 V - 0.4}\right)}, \qquad R_2 = a\sqrt{\left(\frac{\gamma_1 - 1}{\gamma - 1}\right)\left[\frac{0.8 - (\gamma+1)V_2}{(\gamma_1 + 1)V_2 - 0.8}\right]} \quad (5.3)$$

Let us examine the set of integral curves for (1.17) in the region where $P > 0$ (see Section 2). The character of the singular points of (1.17) depends on the values of γ and γ_1.

For example, if $\gamma < 2$, $\gamma_1 > 2$, equation (1.17) has the following singular points.

The point A ($V = 0.4\gamma^{-1}$, $R = 0$) is a node. All integral curves enter this point with an infinite slope. The asymptotic formulas have the form

$$R = C(V - 0.4\gamma^{-1})^{\frac{3}{2\gamma+1}}$$

$$P = \frac{4(\gamma-1)^2 C}{125\gamma^4 G(V)}(V - 0.4\gamma^{-1})^{-\frac{2(\gamma-1)}{2\gamma+1}}$$

$$\lambda = \left(\frac{625K}{8(\gamma-1)^3}\right)^{0.2}\gamma C^{\frac{(\gamma-2)}{5}}(V - 0.4\gamma^{-1})^{\frac{\gamma-1}{2\gamma+1}}[G(V)]^{0.2} \quad (5.4)$$

where

$$G(V) = 1 + \frac{0.4(\gamma-1)(\gamma_1-\gamma)}{\gamma^2(\gamma_1 - 1)}C^2(V - 0.4\gamma^{-1})^{\frac{5-2\gamma}{2\gamma+1}}$$

and C is an arbitrary constant.

The point C ($V = 0.4$, $R = 0$) is a saddle point, the point D ($V = 0.4$, $R = \infty$), a saddle point, the point B($V = 0.4\gamma^{-1}$, $R = \infty$), a saddle point, and the point E($V = 2/(3\gamma_1 - 1)$, $R = \infty$), a node.

Through the point B there passes only the integral curve (2.1); through C and D the straight line $V = 0.4$. All the other integral curves leave point A and enter point E. If $\gamma < 2$ and $\gamma_1 < 2$, the point C is a saddle point, the point D is a node at which all the integral curves meet, and the point E does not belong to the region in which we are interested.

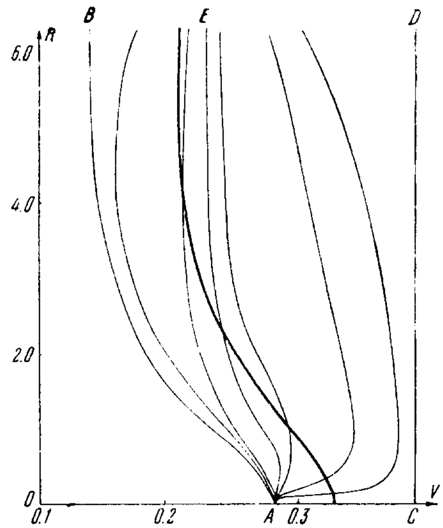

Fig. 5.

If $\gamma > 2$ and $\gamma_1 > 2$, E and C are nodes, while D is a saddle point. In addition, there appears the singular point F ($V = 2/(3\gamma - 1)$, $R = 0$), a saddle point. Through the point F pass the straight line $R = 0$, and the integral curve which separates the integral curves emerging frompppoint A from the curves emerging from point C. All these curves enter point E. Since the character of the singular points A and B does not change in the cases under examination, the formulas (5.4) remain valid.

Finally, if $\gamma_1 \rightarrow \infty$, equation (5.1) takes the form: $\phi(R) = b^2/(a^2+R^2)R$, where a and b are constants.

In the case $\gamma < 2$ the points A and B ($V = 0$, $R = \infty$) are nodes, while C and D are saddle points; when $\gamma > 2$ the points F and D are saddle points, while A, B and C are nodes.

Integral curves of equation (1.17), with $\phi(R)$ defined by (2.1), $\gamma = 1.4$, $\gamma_1 = 3.2$, and $a = 1$, are drawn in Fig.5. The point A, at which $\lambda = 0$, corresponds to the center of symmetry. If γ and $\gamma_1 < 7$, the integral curve which enters point A intersects curve (1.10), which gives the shock condition, at some point (V_2, R_2). From the point ($V = 0$, $R = R_1$) on the R-axis it is possible to get to an integral curve passing through A only by means of a jump at point (V_2, R_2). The motion along an integral curve from point (V_2, R_2) to point A corresponds physically to a motion of the fluid, which can be continued to the center of symmetry. Here, as in the case of a gas, the density and velocity vanish at the center, and the pressure is finite.

When $\gamma > 2$, the point C corresponds to the boundary of the cavity (λ = const, $p = 0$). If $\gamma \geqslant 7$ and $\gamma_1 \geqslant 7$, the integral curves, which give the solution to the problem of a strong blast, correspond to motions with an empty cavity, where the pressure and the density vanish; if $2 < \gamma < 7$, and $\gamma_1 > 7$, or if $2 < \gamma_1 < 7$, and $\gamma > 7$, some of these integral curves correspond to motions which can be continued to the center of symmetry, and some to motions with a cavity.

From formulas (1.20) and (5.1) we get the following expression for the parameter R_1:

$$R_1 = \frac{(\gamma-1)(\gamma_1-1)R_2(R_2^2+a^2)}{(\gamma-1)(\gamma_1+1)R_2{}^2+(\gamma_1-1)(\gamma+1)a^2} \qquad (5.5)$$

Consequently, the parameter R_1, belonging to the integral curves which give the solution to the problem of a strong blast, varies from zero to infinity.

For the value $R_1 = 0$, we get a solution of (2.3), which corresponds to the limiting case of an incompressible fluid with vanishing density and pressure. In this case, as for a gas with adiabatic exponent γ, $R_2/R_1 = (\gamma + 1)/(\gamma - 1)$. From (5.5) it follows that the ratio R_2/R_1 varies as R_1 increases, taking on its minimum value $(\gamma_1 + 1)/(\gamma_1 - 1)$ when the density in front of the shock front is infinite, corresponding to which the density behind the front also tends to infinity, while the speed of the front and of the particles behind it goes to zero.

In the neighborhood of the point A, the asymptotic formulas (5.4) for R, P, and λ coincide, except for multiplicative constants, with the analogous formulas for a gas with adiabatic exponent γ. Since $R_2/R_1 = (\gamma + 1)/(\gamma - 1)$ only for $R_1 = 0$, the behavior of the functions v/v_2, ρ/ρ_2, and p/p_2 near $\lambda = 0$ and $\lambda = 1$ for any non-vanishing R_1, differs quantitatively somewhat from the behavior of these functions for gases with adiabatic exponents γ and γ_1.

On Fig.6 are shown the graphs of v/v_2 and ρ/ρ_2 as functions of λ, for an initial value of $R_1 = 1.862$. The constant α in the energy formula turned out to have the value 0.533. For $\lambda = 0$ $p/p_2 = 0.3444$, while for $\lambda = 0.5204$ p/p_2 is at its minimum, and is equal to 0.2703.

6. It is easy to see that the case $\phi(R) = a$ = const can be reduced to the case $\phi(R) = 1$.

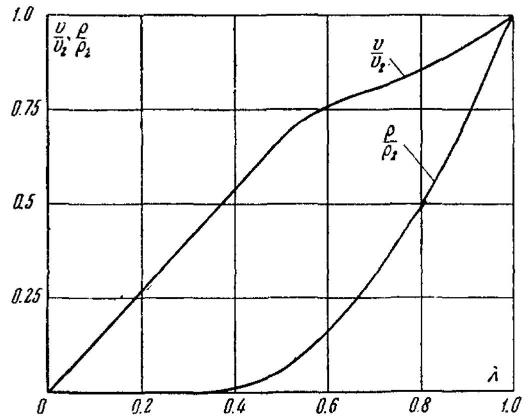

Fig. 6.

The exact solution of (2.1) and the shock condition (1.10) are given in this case by the formulas

$$R = \frac{V}{0.4 - V}, \qquad R_2 = \frac{V_2}{2(0.4 - V_2)} \tag{6.1}$$

From the equation (3.1) it is clear in this case that there is no real sonic line. The set of integral curves for the case $\phi(R) = 1$ given in Fig. 7.

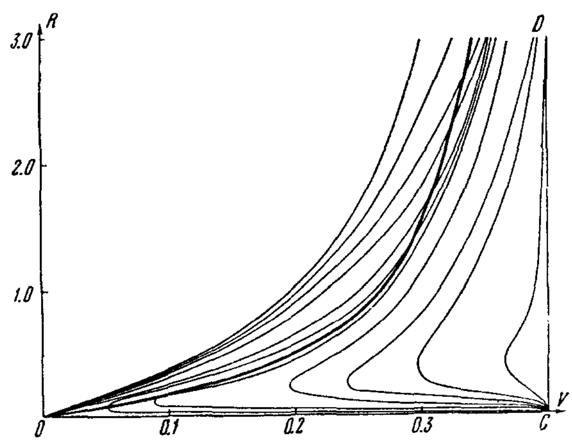

Fig. 7.

In the neighborhood of the singular point O ($V = 0$, $R = 0$) the equation (1.17) can be written in the following form:

$$\frac{dR}{dV}=\frac{-2R^2(5V+R)}{15V^2-16VR+4R^2} \tag{6.2}$$

This is a compound singularity, which consists of a saddle point and a node.

The point C ($V=0.4$, $R=0$) is a node. The asymptotic formulas for V, P, and λ in the neighborhood of C have the form:

$$V=0.4-\frac{C_1}{R}\exp\left(-\frac{1}{2R}\right),\qquad P=0.2C_1\exp\left(-\frac{1}{2R}\right),\qquad \lambda=\left(\frac{5K}{C_1^2}\right)^{0.2} \tag{6.3}$$

where C_1 is an arbitrary constant. Thus, this point corresponds to the boundary of the cavity.

The point D ($V=0.4$, $R=\infty$) is a node, at which all the integral curves leaving C meet. At the point ($V=1/10$, $R=1/6$) one of these curves is tangent to the second of curves (6.1); curves with a large value of the constant C_1, which enters formula (6.3), cross this curve at two points. From the point ($V=0$, $R=R_1$) it is possible to reach the integral curve which enters the point O, corresponding to the center of symmetry, or the point C, corresponding to the boundary of the cavity, only by means of a jump.

The parameter R_1 varies from 0 to 0.5.

As in the previous case (Section 5), the value $R_1=0$ corresponds to the solution (2.3).

The pressure and the density vanish at every point of the integral curve; the shock wave immediately goes to infinity.

If $R_1=0.5$, the density behind the shock front becomes infinite, while the speed of the shock wave goes to zero.

Fig.8 shows the graphs of v/v_2, ρ/ρ_2, and p/p_2 as functions of λ. The curves 1, 2 and 3, for the velocity, density and pressure respectively, correspond to the value $R_1=0.125$, curves 4, 5 and 6 to $R_1=0.375$. The value $R_1=0.125$ corresponds to the integral curve which, at the point ($V=1/10$, $R=1/6$), is tangent to the second of the curves (6.1), and which enters point C. There results a motion with a cavity, expanding from the center of symmetry, on the boundary of which both the pressure and the density vanish.

In the second case the solution can be continued to the center of symmetry, where the velocity, density and pressure all vanish.

The constant α has the values 0.00274 and 0.05502 for the first and second case, respectively.

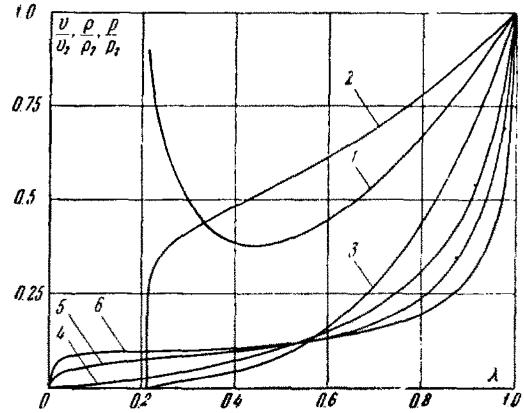

Fig. 8.

7. In the general case of a point blast, the boundary conditions contain the three-dimensional parameters ρ_0, E_0, and the initial pressure, p_1, of the undisturbed medium. If this pressure, p_1, is sufficiently small to be neglected, we obtain a problem with self-similar solutions.

We will now try to estimate more precisely the distance which a shock wave can travel before the self-similar solution becomes inaccurate.

Instead of (1.6) let us write the exact shock conditions:

$$-\rho_1 c = \rho_2 (v_2 - c), \qquad \rho_1 c^2 + p_1 = \rho_2 (v_2 - c)^2 + p_2$$
$$\frac{1}{2} c^2 + \frac{p_1}{\rho_1} + \frac{p_1}{\rho_0} \varphi\left(\frac{\rho_1}{\rho_0}\right) = \frac{1}{2} (v_2 - c)^2 + \frac{p_2}{\rho_2} + \frac{p_2}{\rho_0} \varphi\left(\frac{\rho_2}{\rho_0}\right) \tag{7.1}$$

Eliminating p_1 from these equations by means of $p_1 = \rho_0 a_1^2 \chi(R_1)/\chi'(R_1)$ and introducing a new variable $q = a_1^2/c^2$, where a_1 is the sound speed in the undisturbed medium, while $\chi(R)$ is determined from (1.3), we will get the following expressions for the density, velocity, and pressure as functions of the shock speed, c, the quantity q, the density ρ_1, in front of the shock wave, and the density ρ_0 :

$$\chi'\left(\frac{\rho_1}{\rho_0}\right)\left(1 - \frac{\rho_1}{\rho_2}\right)\left\{\frac{1}{2}\left(1 - \frac{\rho_1}{\rho_2}\right) - \frac{\rho_1}{\rho_0} \varphi\left(\frac{\rho_2}{\rho_0}\right)\right\} =$$
$$= \chi\left(\frac{\rho_1}{\rho_0}\right)\left\{\varphi\left(\frac{\rho_2}{\rho_0}\right) + \frac{\rho_0}{\rho_2} - \varphi\left(\frac{\rho_1}{\rho_0}\right) - \frac{\rho_0}{\rho_1}\right\} q \tag{7.2}$$
$$v_2 = c\left(1 - \frac{\rho_1}{\rho_2}\right), \qquad p_2 = \rho_0 c^2 \left\{\frac{\rho_1}{\rho_0}\left(1 - \frac{\rho_1}{\rho_2}\right) + q \frac{\chi(\rho_1/\rho_0)}{\chi'(\rho_1/\rho_0)}\right\}$$

Making use of (7.2) for a given function $\phi(R)$ and a given value of R_1, we can find a value $q = q_0$, for which the relative error in computing the characteristics of the motion on the shock wave, v_2, ρ_2, p_2, by means of the formulas (1.21) and (7.2), is, for example, 0.05. Consequently, in solving the problem of a blast for $q \leqslant q_0$, the shock conditions can be taken in the form (1.21), i.e. the self-similar solutions can be used for values of r_2 not exceeding

$$\omega(R_1)\left(\frac{E_0}{p_1}\right)^{1/\nu} \qquad \left(\omega(R_1) = \left[\frac{4q_0}{(2+\nu)^2}\frac{\chi(R_1)}{\chi'(R_1)\,\alpha(R_1)}\right]^{1/\nu}\right)$$

For the value of R_1 examined in Section 4, the quantity q_0 has the value 0.072.

We feel it our duty to express our deep gratitude to L.I. Sedov for his direction of this work.

BIBLIOGRAPHY

1. Sedov, L.I., *Metody podobiia i razmernosti v mekhanike (Methods of Similarity and Dimensional Analysis in Mechanics)*. Gostekhizdat, 1957.
2. Sedov, L.I., Dvizhenie vozdukha pri sil'nom vzryve (Motion of the air with a strong blast). *Dokl. Akad. Nauk SSSR* Vol.52, No.1, 1946.
3. Sedov, L.I., Raspostranenie sil'nykh vzryvnykh voln (Propagation of strong shock waves). *PMM* Vol.10, No.2, 1946.
4. Bam-Zelikovich, G.M., Raspostranenie sil'nykh vzrryvnykh voln (Propagation of strong shock waves). *Collection No.4 Theoretical Hydromechanics* (Edited by L.I. Sedov). Oborongiz, 1949.
5. Cole, P., *Podvodnye vzryvy (Underwater Explosions). Izd. inostr.lit.* Moscow, 1950.
6. Bridgman, Freezing parameters and compressions of twenty-one substances to 50,000 kg/cm. *Proc. Am. Acad. Sci.* Vol.74, No.12, 1942.
7. Bridgman, The phase diagram of water to 45,000 kg/cm. *J. Chem. Phys.* Vol.5, No.10, 1937.

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